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GROUP STRATIFICATION AND EXACT SOLUTIONS OF THE EQUATION OF TRANSONIC GAS MOTIONS

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Exact solutions of the Kármán–Guderley equation that describes spatial gas flows in the transonic approximation are considered. A group stratification of the equation with respect to the infinite-dimensional part of the admissible group is constructed. New invariant and partly invariant solutions are obtained. The possibility of existence of solutions continuous in the entire space is analyzed for invariant submodels with one independent variable. A solution of the Kármán–Guderley equation of the double-wave type is constructed.

Key words: *transonic flow, invariant solutions, double waves, group stratification.*

1. Formulation of the Problem. One of the widespread models used to describe transonic gas flows is the Kármán–Guderley equation

$$-\varphi_x\varphi_{xx} + \varphi_{yy} + \varphi_{zz} = 0. \qquad (1.1)$$

This equation describes small perturbations of a gas flow moving with a critical velocity along the Ox axis. The derivation of this equation and examples of solving particular gas-dynamic problems can be found in [1–3]. The case of two independent variables is considered most comprehensively. In this case, Eq. (1.1) is equivalent to a system of two first-order equations, which is linearized by the hodograph transformation. The group properties and exact solutions of such a system were studied in [4]. In the spatial case, self-similar solutions of Eq. (1.1) were mainly considered. The application of group-analysis methods to the Kármán–Guderley equation allows one to enlarge the list of its exact solutions.

In classifying the solutions, it turned out to be useful to construct a group stratification of Eq. (1.1) with respect to the infinite-dimensional part of the admissible group. Based on the group stratification, all invariant submodels with one and two independent variables are obtained and given. For submodels where the invariant independent variable depends linearly on the polar angle, the possibility of existence of solutions continuous in the entire space is analyzed. A partially invariant solution of Eq. (1.1) of the double-wave type is constructed, which possesses arbitrariness in four functions of one argument. The remaining partially invariant solutions can be obtained by using the optimal system of subgroups for the finite-dimensional part of the admissible group, which is constructed in the present work.

2. Group Properties of the Kármán–Guderley Equation. It is shown by direct calculations that Eq. (1.1) admits an infinite-dimensional algebra of transformations $L_6 \oplus L_\infty$. Its finite-dimensional part L_6 is generated by the following operators:

$$\begin{gathered} Y_1 = \partial_x, \quad Y_2 = \partial_y, \quad Y_3 = \partial_z, \quad Y_4 = z\partial_y - y\partial_z, \\ Y_5 = y\partial_y + z\partial_z - 2\varphi\partial_\varphi, \qquad Y_6 = x\partial_x + 3\varphi\partial_\varphi. \end{gathered} \qquad (2.1)$$

The infinite-dimensional part L_∞ of the admissible algebra corresponds to the operator

$$X_\infty = f(y,z)\partial_\varphi, \qquad \Delta f(y,z) = 0. \qquad (2.2)$$

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There are no nontrivial contact transformations admitted by Eq. (1.1). The objective of the present work is to construct group-invariant solutions of Eq. (1.1) on the basis of the admissible group corresponding to the Lie algebra $L_6 \oplus L_\infty$. The infinite dimensionality of the admissible group of transformations does not allow a constructive description of the set of classes of its conjugate subgroups. To solve this problem, we construct a group stratification of Eq. (1.1) with respect to the infinite-dimensional part L_∞ of the admissible group. This transformation of the Kármán–Guderley equation allows its representation as an equivalent unification of two systems: automorphic and resolving ones. The automorphic system has the following property: two arbitrary solutions of this system lie on the orbit of one of them, i.e., one it transformed to the other by the group transformation. On the contrary, the resolving system does not admit a group for constructing the stratification and, thus, "distinguishes" the orbits of essentially different solutions. The resolving system inherits only the finite-dimensional part of the initial admissible group, and its solutions are classified on the basis of the known algorithm [5]. The solutions of the initial equation are reconstructed by integration of the automorphic system. Detailed information on automorphic systems of differential equations and the algorithm of constructing the group stratification with respect to a specified group can be found in [6].

The first stage of constructing the group stratification is the calculation of the basis of differential invariants for the transformation generated by the algebra L_∞.

Lemma 1. *The basis of differential invariants for the transformation generated by the operator X_∞ can be chosen in the form*

$$x, \; y, \; z, \; \varphi_x, \; \varphi_{yy} + \varphi_{zz}. \tag{2.3}$$

Operators of invariant differentiation are operators of total differentiation with respect to the independent variables x, y, and z.

Proof. Lemma 1 is proved as follows. First, it is shown that functions (2.3) are invariants of the prolonged operator X_∞. Then, calculating the dimensionalities of the prolonged space and the rank of the prolonged group, one can prove that differentiation of functions (2.3) with respect to independent variables can yield any invariant of an arbitrarily high order.

Lemma 2. *The group stratification of Eq.* (1.1) *is specified by the automorphic system consisting of two equations*

$$\varphi_x = a, \qquad \varphi_{yy} + \varphi_{zz} = a a_x \tag{2.4}$$

and the resolving equation determining the function $a(x, y, z)$:

$$-a a_{xx} - a_x^2 + a_{yy} + a_{zz} = 0. \tag{2.5}$$

Proof. To construct the group stratification, one should assign two invariants of the basis as functions of the remaining three invariants. In this case, the invariants x, y, and z are chosen as independent variables. With allowance for the initial equation (1.1), we obtain system (2.4). The only condition of its compatibility is Eq. (2.5), which defines the resolving part of the group stratification. The automorphy of system (2.4) (as a system for the function φ with a specified function a) follows from the fact that the arbitrariness in the solution of its first equation is an additive function depending on y and z, and the arbitrariness in the solution of the second equation is an additive function of the variables x, y, and z, harmonic in terms of y and z. Thus, the total arbitrariness in the solution of system (2.4) is an additive harmonic function of the variables y and z. This arbitrariness is exhausted by the transformation generated by the operator X_∞.

The calculation of the group of contact transformations admitted by Eq. (2.5) shows that it admits only the finite-dimensional Lie algebra L_6 isomorphic to algebra (2.1). The basis of its operators can be chosen in the form

$$\begin{gathered} X_1 = \partial_x, \quad X_2 = \partial_y, \quad X_3 = \partial_z, \quad X_4 = z\partial_y - y\partial_z, \\ X_5 = y\partial_y + z\partial_z - 2a\partial_a, \qquad X_6 = x\partial_x + 2a\partial_a. \end{gathered} \tag{2.6}$$

The finite dimensionality of algebra (2.6) allows one to construct its optimal system of subalgebras and give a complete description of group-invvariant solutions of Eq. (2.5). The initial function φ is restored by integration of the involutive system (2.4) with a known function $a(x, y, z)$.

3. Optimal System of Subalgebras. To construct the optimal system of subalgebras ΘL_6, we use a two-step algorithm based on the composition series of algebra ideals [5]. The commutator relations of algebra (2.6) are listed in Table 1.

TABLE 1

Operator number	X_1	X_2	X_3	X_4	X_5	X_6
X_1	0	0	0	0	0	X_1
X_2	0	0	0	$-X_3$	X_2	0
X_3	0	0	0	X_2	X_3	0
X_4	0	X_3	$-X_2$	0	0	0
X_5	0	$-X_2$	$-X_3$	0	0	0
X_6	$-X_1$	0	0	0	0	0

TABLE 2

A_i	$\bar{x}^1$	$\bar{x}^2$	$\bar{x}^3$
A_1	$x^1 - \alpha_1 x^6$	x^2	x^3
A_2	x^1	$x^2 - \alpha_2 x^5$	$x^3 + \alpha_2 x^4$
A_3	x^1	$x^2 - \alpha_3 x^4$	$x^3 - \alpha_3 x^5$
A_4	x^1	$x^2 \cos\alpha_4 + x^3 \sin\alpha_4$	$-x^2 \sin\alpha_4 + x^3 \cos\alpha_4$
A_5	x^1	$\alpha_5 x^2$	$\alpha_5 x^3$
A_6	$\alpha_6 x^1$	x^2	x^3

The group of inner automorphisms $\operatorname{Int} L_6$ is generated by a set of one-parameter groups A_i constructed for each basis vector X_i. Their action on an arbitrary element $X = x^i X_i \in L_6$ is described by the action of the corresponding matrices $A_i(\alpha_i) = \exp(\alpha_i \operatorname{ad} X_i)$ on the vector-column $\boldsymbol{x} = (x^i)$ (Table 2). The coordinates x^4, x^5, x^6 are invariants of the group of inner automorphisms. By virtue of the involutions $x \to -x$ and $(y, z) \to -(y, z)$, we can assume that the parameters α_5 and α_6 take both positive and negative values. The following composition series of ideals is used:

$$\{X_1\} \subset \{X_1, X_2, X_3\} \subset \{X_1, X_2, X_3, X_4\} \subset \{X_1, X_2, X_3, X_4, X_5\} \subset L_6. \tag{3.1}$$

At each step of this series, the ideal–subalgebra decomposition is possible.

In the coordinate representation, each r-dimensional algebra $M \subset L_6$ corresponds to the $(r \times 6)$ matrix $M = \{\xi^i_j\}$, whose rows contain coordinates of its basis elements. Between the rows $H_j = (\xi^1_j, \ldots, \xi^6_j)$, the subalgebra conditions $[H_p, H_q] = K^s_{pq} H_s$ $(p, q = 1, \ldots, r)$ should be satisfied, where the commutator $[H_p, H_q]$ is calculated by the formula $[H_p, H_q]^l = C^l_{ij} \xi^i_p \xi^j_q$ [C^l_{ij} is the structural tensor of the algebra L_6 (see Table 1) and $K^s_{p,q}$ is the structural tensor of the algebra M]. The basis transformations (arbitrary transformations of rows) and inner automorphisms (transformations of columns determined in Table 2) act on the set of matrices M. The construction of the optimal system of subalgebras reduces to enumeration of all matrices M satisfying the subalgebra conditions with accuracy to the transformations mentioned above.

In accordance with the composition series (3.1), the optimal system ΘL_6 is constructed as follows: we use the basic decomposition $L_6 = J \oplus N$ with the ideal $J = \{X_1, X_2, X_3\}$ and subalgebra $N = \{X_4, X_5, X_6\}$. First, we construct the optimal systems ΘN and ΘJ.

Since the operators X_4, X_5, and X_6 are invariants of the group $\operatorname{Int} L_6$, the subalgebras N are classified in terms of the basis transformations only. The optimal system ΘN can be chosen in the form

$$\begin{gathered} \{X_4,\ X_5,\ X_6\}, \\ \{X_5 + \alpha X_6,\ X_4 + \beta X_6\}, \quad \{X_4 + \alpha X_5,\ X_6\}, \quad \{X_5,\ X_6\}, \\ \{X_4 + \alpha X_5 + \beta X_6\}, \quad \{X_5 + \alpha X_6\}, \quad \{X_6\}. \end{gathered} \tag{3.2}$$

To construct the optimal system ΘJ, we can use all transformations $\operatorname{Int} L_6$. As a result, we obtain

$$\begin{gathered} \{X_1,\ X_2,\ X_3\}, \\ \{X_1 + X_2,\ X_3\}, \quad \{X_1,\ X_3\}, \quad \{X_2,\ X_3\}, \\ \{X_1 + X_2\}, \quad \{X_1\}, \quad \{X_3\}. \end{gathered} \tag{3.3}$$

At the last stage, the optimal systems (3.2) and (3.3) are "combined" in a usual manner. The subalgebras L_6 are described by $(r\times 6)$ matrices $(r = 1, \dots, 6)$, which can be always brought to the block form by basis transformations:

$$M = \begin{pmatrix} \xi & \eta \\ \zeta & 0 \end{pmatrix}.$$

Here, the blocks ξ, η, and ζ are matrices containing three columns each. The block η corresponds to the representatives ΘN, the block ζ corresponds to the subalgebras of J, and the block ξ corresponds to the matrix with indeterminate coefficients. After substitution of particular elements $J_k \in \Theta J$ and $N_l \in \Theta N$ into the matrix M, the block ξ is simplified to the maximum extent by means of basis transformations, which do not change the block η, and inner automorphisms. For rows of the matrix M, we verify the subalgebra conditions that impose additional restrictions on arbitrary elements remaining in the block ξ. The optimal system obtained should be normalized, i.e., the arbitrariness in choosing representatives of equivalence classes should be used so that, in addition to each subalgebra M, the optimal system ΘL_6 contained its normalizer $\mathrm{Nor}_{L_6} M$.

The normalized optimal system of subalgebras ΘL_6 is given in Sec. 7.

4. Invariant Submodels. To construct invariant submodels, we use one-dimensional and two-dimensional representatives of the optimal system ΘL_6. The invariant representation for the function φ is obtained as follows. For each representative $H \subset \Theta L_6$, the conditions of existence of the invariant solution of Eq. (2.5) are verified. The representation for the function a is written, from which the representation for φ is restored by integration of the automorphic system (2.4). We demonstrate the algorithm described on a submodel that defines gas flows with helical level surfaces.

Submodel 2.4. The generating subalgebra is defined by the operators $L_{2.4} = \{X_1 + X_4,\, X_5\}$. In the cylindrical coordinate system (x, r, θ), the basis operators of the subalgebra are written as

$$L_{2.4} = \{\partial_x + \partial_\theta,\ r\partial_r - 2a\partial_a - 2\varphi\partial_\varphi\}$$

[here, representations (2.1) and (2.6) are united]. The invariants of $L_{2.4}$ are

$$\lambda = x - \theta, \quad r^2\varphi, \quad r^2 a. \tag{4.1}$$

The representation of the solution is $a = r^{-2}b(\lambda)$. Substituting the solution representation into Eq. (2.5), we obtain the equation for the invariant function $b(\lambda)$:

$$(b - 1)b'' + b'^2 - 4b = 0.$$

We restore the function φ by integrating the automorphic system (2.4) whose first equation yields

$$\varphi = \frac{1}{r^2}\int b(\lambda)\,dx = \frac{1}{r^2}\int b(\lambda)\,d\lambda = \frac{1}{r^2}\,B(\lambda) + \varphi^0(r,\theta). \tag{4.2}$$

Here $b(\lambda) = B'(\lambda)$ and φ^0 is a certain function, which is arbitrary at the moment. We substitute representation (4.2) into the second equation of (2.4). Grouping the terms depending on different variables, we obtain the equation

$$\frac{1}{r^4}(B'' - B'B'' + 4B) + \varphi^0_{rr} + \frac{1}{r}\,\varphi^0_r + \frac{1}{r^2}\,\varphi^0_{\theta\theta} = 0. \tag{4.3}$$

Separating variables in (4.3), we find

$$(1 - B')B'' + 4B = C, \quad \varphi^0_{rr} + \frac{1}{r}\,\varphi^0_r + \frac{1}{r^2}\,\varphi^0_{\theta\theta} = -\frac{C}{r^4}, \quad C = \text{const}. \tag{4.4}$$

By virtue of automorphism of system (2.4), any two solutions of this system are related by the group transformation (2.2); therefore, to find the function φ^0, it is sufficient to find any particular solution of the second equation of (4.4). For instance, we can take the solution

$$\varphi^0 = -C/(4r^2). \tag{4.5}$$

A comparison of (4.2) and (4.5) shows that, with accuracy to the choice of the function $B(\lambda)$, the representation of the function φ is equivalent to the representation $\varphi = r^{-2}B(\lambda)$. The same representation can be obtained by constructing the invariant $L_{2.4}$-solution for the initial equation (1.1).

For most representatives $H \subset \Theta L_6$, the invariant submodels obtained by the method described above coincide with the H-invariant solutions of Eq. (1.1). This is invalid for submodels 2.8, 2.18, and 2.19. In these cases, the algorithm related to integration of the resolving equation and automorphic system yields a wider class of solutions.

TABLE 3

Submodel number	Solution representation	Submodel equations
2.1	$\varphi = x^3 r^{-2} B(\lambda)$, $\lambda = \alpha\theta + \beta \ln r - \ln\|x\|$	$(\alpha^2 + \beta^2 - 3B + B')B'' - 5(B')^2 +$ $+ (-4\beta + 21B)B' + 4B - 18B^2 = 0$
2.2	$\varphi = r^{-2}B(\lambda)$, $\lambda = x$	$B'B'' = 4B$
2.3	$\varphi = r^{-2}B(\lambda)$, $\lambda = x - \alpha\theta - \ln r$	$(1 + \alpha^2 - B')B'' + 4B' + 4B = 0$
2.4	$\varphi = r^{-2}B(\lambda)$, $\quad \lambda = x - \theta$	$(1 - B')B'' + 4B = 0$
2.5	$\varphi = x^3 r^{-2} B(\lambda)$, $\lambda = \alpha\theta - \ln r$	$(1 + \alpha^2)B'' + 4B' + 4B - 18B^2 = 0$
2.6	$\varphi = x^3 r^{-2} B(\lambda)$, $\quad \lambda = \theta$	$B'' + 4B - 18B^2 = 0$
2.8	$\varphi = z(B(\lambda) + C \ln\|z\|)$, $\lambda = (x - y)/z$	$B''(B' - 1 - \lambda^2) = C$
2.9	$\varphi = x^3 y^{-2} B(\lambda)$, $\quad \lambda = x\, y^{-\alpha}$	$(\alpha^2 - 3B - \lambda B')\lambda^2 B'' + (5\alpha + \alpha^2 -$ $- 24B)\lambda B' - 6\lambda^2 (B')^2 - 18B^2 + 6B = 0$
2.10	$\varphi = y^{-2}B(\lambda)$, $\lambda = x$	$B'B'' = 6B$
2.11	$\varphi = y^{-2}B(\lambda)$, $\quad \lambda = x - \ln\|y\|$	$(1 - B')B'' + 5B' + 6B = 0$
2.14	$\varphi = x^3 B(\lambda)$, $\lambda = y$	$B'' = 18B^2$
2.15	$\varphi = x^3 B(\lambda)$, $\quad \lambda = y - \ln\|x\|$	$(1 - 3B + B')B'' - 5(B')^2 + 21BB' - 18B^2 = 0$
2.18	$\lambda = x - z$	$\varphi = x + \operatorname{sign}(x - y)\, y^2/4 \pm 2\|x - y\|^{3/2}/3$
2.19	$\lambda = z$	$\varphi = \pm\|x\|^{3/2} + 9 \operatorname{sign}(x)\, y^2/16$

All invariant submodels depending on one and two independent variables are listed in Tables 3 and 4. The first column shows the submodel numbers in accordance with the numbers of the generating subalgebras in ΘL_6. The second column contains the solution representation. The invariant function B depends on the invariant independent variables λ and μ in the case of submodels of rank 2 and on one variable λ in the case of submodels of rank 1. The third column gives the equation for determining the function B. In most submodels, the invariant independent variable is chosen such that the resultant equation for the function B becomes autonomous. Only submodels in which the function φ depends on the variable x are given.

5. Submodels of Rank 1. In clarifying the character of motion described by invariant submodels, the major role belongs to the shape of the level surfaces $\lambda = \text{const}$. Important characteristics of the flow, such as the sonic surfaces, shock-wave shape, invariant characteristics, and limiting surface, are given by the equation $\lambda = C$ with a certain constant C determined from additional conditions. It follows from Table 3 that the shape of level surfaces can be quite nontrivial.

In submodels 2.1, 2.3, 2.4, 2.5, and 2.6, the invariant independent variable λ depends linearly on the polar angle θ. This circumstance imposes a restriction on the possible form of the function B: for the solution to be continuous in the entire space, the function B should be periodic with the half-period

$$T = \pi/N, \qquad N \in \mathbb{N}. \tag{5.1}$$

An analysis of singular points on the phase plane (B', B) for all the submodels mentioned shows that the singular points of the "center" type are observed only in submodels 2.4 and 2.6. Let us verify the possibility of satisfying condition (5.1) for them.

Submodel 2.4. The solution representation in the cylindrical coordinate system has the form

$$\varphi = B(\lambda)/r^2, \qquad \lambda = x - \theta. \tag{5.2}$$

The function $B(\lambda)$ satisfies the equation

$$(1 - B')B'' + 4B = 0. \tag{5.3}$$

TABLE 4

Submodel number	Solution representation	Submodel equation
1.1	$\varphi = r^{-2}B(\lambda,\mu)$, $\lambda = \alpha\theta - \ln r$, $\mu = x - \theta$	$(1+\alpha^2)B_{\lambda\lambda} + (1 - B_\mu)B_{\mu\mu} - 2\alpha B_{\lambda\mu} + {}$ $+ 4B_\lambda + 4B = 0$
1.2	$\varphi = x^3 r^{-2}B(\lambda,\mu)$, $\lambda = \alpha\theta - \ln r$, $\mu = \ln r - k\ln\lvert x\rvert$, $k = \alpha/\beta$	$(1+\alpha^2)B_{\lambda\lambda} + (1 - 3k^2B + k^3B_\mu)\,B_{\mu\mu} - 2\,B_{\lambda\mu} + {}$ $+ 4B_\lambda + (21kB - 4)B_\mu - 5k^2B_\mu^2 - 18B^2 + 4B = 0$
1.3	$\varphi = r^{-2}B(\lambda,\mu)$, $\lambda = \alpha\theta - \ln r$, $\mu = x$	$(1+\alpha^2)B_{\lambda\lambda} - B_\mu B_{\mu\mu} + 4B_\lambda + 4B = 0$
1.4	$\varphi = r^{-2}B(\lambda,\mu)$, $\lambda = \theta$, $\mu = x - \ln r$	$B_{\lambda\lambda} + (1 - B_\mu)B_{\mu\mu} + 4B_\mu + 4B = 0$
1.5	$\varphi = x^3 r^{-2}B(\lambda,\mu)$, $\lambda = \theta$, $\mu = \ln\lvert x\rvert - \alpha\ln r$	$B_{\lambda\lambda} + (\alpha^2 - 3B - B_\mu)B_{\mu\mu} - 5B_\mu^2 + {}$ $+ (4\alpha - 21B)B_\mu - 18B^2 + 4B = 0$
1.6	$\varphi = r^{-2}B(\lambda,\mu)$, $\lambda = \theta$, $\mu = x$	$B_{\lambda\lambda} - B_\mu\,B_{\mu\mu} + 4B = 0$
1.7	$\varphi = x^3B(\lambda,\mu)$, $\lambda = z$, $\mu = y - \ln\lvert x\rvert$	$B_{\lambda\lambda} + (1 - 3B + B_\mu)B_{\mu\mu} - {}$ $- 5B_\mu^2 + 21B\,B_\mu = 18B^2$
1.8	$\varphi = x^3B(\lambda,\mu)$, $\lambda = x$, $\mu = y$	$B_{\lambda\lambda} + B_{\mu\mu} = 18B^2$
1.9	$\varphi = B(\lambda,\mu)$, $\lambda = x - y$, $\mu = x$	$(1 - B_\lambda)\,B_{\lambda\lambda} + B_{\mu\mu} = 0$
1.10	$\varphi = B(\lambda,\mu)$, $\lambda = x$, $\mu = y$	$B_\lambda\,B_{\lambda\lambda} - B_{\mu\mu} = 0$

In the parametric form, the general solution of Eq. (5.3) is defined by the formulas

$$\lambda = \mp\frac{\sqrt{3}}{2}\int\frac{(p+1)\,dp}{\sqrt{C+2p^3-3p^2}}, \qquad B = \pm\frac{1}{2\sqrt{3}}\sqrt{C+2p^3-3p^2}$$

and can be represented in terms of elliptic functions. Still, it is convenient to describe the behavior of the function $B(\lambda)$ using the pattern of integral curves on the phase plane (p, B), $p = B'$, which are specified by the equation

$$-2p^3 + 3p^2 + 12B^2 = C. \tag{5.4}$$

According to Newton's classification, curve (5.4) is a diverging parabola. Curves (5.4) for different values of C are plotted in Fig. 1. There are two critical values $C = 0$ and $C = 1$. In the first case, curve (5.4) is a curve (dashed curve in Fig. 1) and the point $(0, 0)$. For $0 < C < 1$, the point "grows up" into closed curves. For $C = 1$, the curves on the right and the closed curves around the point $(0, 0)$ merge, forming a loop (bold curve in Fig. 1). The periodic solutions of Eq. (5.3) correspond to the values $0 \leqslant C \leqslant 1$. The half-period of the solution is

$$T = \frac{\sqrt{3}}{2}\int\limits_{p_1}^{p_2}\frac{(p+1)\,dp}{\sqrt{C+2p^3-3p^2}}, \tag{5.5}$$

where p_i are the roots of the equation $C = -2p^3 + 3p^2$ arranged in ascending order (Fig. 2).

We consider the limiting cases $C \to 0$ and $C \to 1$. In the first case, we have p_1, $p_2 \to 0$, and $p_3 \to 3/2$. Performing in (5.5) the limiting transition as $C \to 0$, we obtain

$$T = \frac{\sqrt{3}}{2\sqrt{2}}\int\limits_{p_1}^{p_2}\frac{p+1}{\sqrt{p_3-p}}\,\frac{dp}{\sqrt{(p_2-p)(p-p_1)}} \underset{C\to 0}{\longrightarrow} \frac{\pi}{2}.$$

As $C \to 1$, the solution of Eq. (5.3) is calculated explicitly:

$$B = \lambda(4\lambda^2/9 - 1)/2.$$

Its half-period is $T = 3/2$. For intermediate values $0 < C < 1$, the function $T(C)$ monotonically decreases. Thus, we obtain

$$3/2 \leqslant T \leqslant \pi/2. \tag{5.6}$$

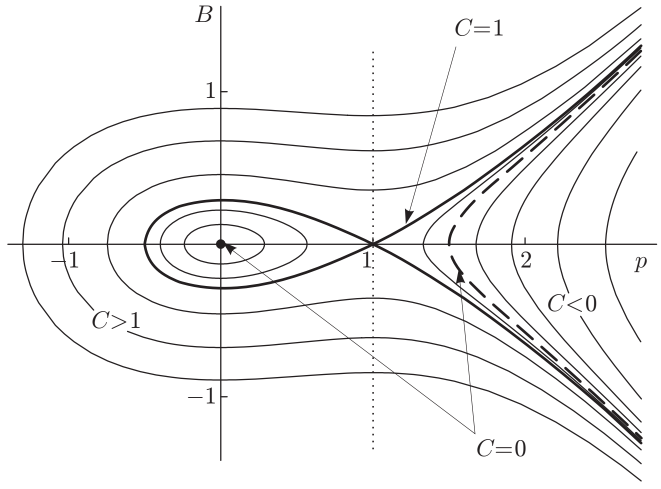

Fig. 1

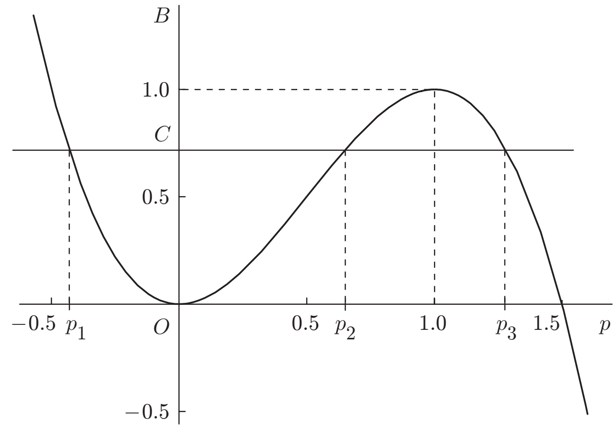

Fig. 2

For the solution to be continuous in the entire space, equality (5.1) should be satisfied. It follows from (5.1) and (5.6) that $N = 2$. But then we have $C = 0$, which corresponds to $\varphi \equiv 0$. Thus, there are no nontrivial motions of the gas of the form (5.2), which are continuous in the entire space.

Submodel 2.6. The solution representation has the form $\varphi = x^3 r^{-2} B(\theta)$. The submodel equation is

$$B'' + 4B + 18B^2 = 0.$$

The equation of the curves on the phase plane (p, B) is

$$p^2 + 4B^2 + 12B^3 = 4C.$$

These curves are also diverging parabolas. The closed curves correspond to the values $0 < C < 4/243$. The half-period of the solution is

$$T = \frac{1}{2} \int_{B_2}^{B_3} \frac{dB}{\sqrt{C - 3B^3 - B^2}}.$$

Here B_i are the roots of the equation $3B^3 + B^2 = C$, arranged in ascending order. The limiting values are $T \to \pi/2$ for $C \to 0$ and $T \to +\infty$ for $C \to 4/243$. For $0 < C < 4/243$, the half-period T monotonically increases. Thus, we obtain

$$\pi/2 \leqslant T \leqslant +\infty.$$

There exist two values of T satisfying (5.1) for $N = 1$ and $N = 2$. The case $N = 2$ corresponds to the trivial solution $\varphi \equiv 0$. Thus, the only nontrivial solution continuous in the entire space is the 2π-periodic solution obtained for $C \approx 0.016436$.

6. Multiple Waves. Solutions with a degenerate hodograph, or multiple waves, from the viewpoint of the group analysis of differential equations, are irregular partially invariant solutions constructed on the full group of translations of the space of independent variables [6]. For gas-dynamic equations, a large number of solutions of the multiple-wave type are given in [7]. The case of potential gas flows is studied rather comprehensively. The solutions of the multiple-wave type for Eq. (1.1) are described below.

To obtain partially invariant solutions, we have to consider the first prolongation of the basic space. We introduce the notation

$$u = \varphi_x, \qquad v = \varphi_y, \qquad w = \varphi_z, \tag{6.1}$$

in Eq. (1.1) and

$$u = a_x, \qquad v = a_y, \qquad w = a_z. \tag{6.2}$$

in Eq. (2.5). In both cases, the following equality should be satisfied:

$$\operatorname{rot} \boldsymbol{u} = 0. \tag{6.3}$$

6.1. *Simple Wave.* The solution of the simple-wave type for the equation of transonic gas motions (1.1) is classical. In the spatial case, it is defined by the relations

$$v = v(u), \qquad w = w(u), \qquad u = u(x, y, z). \tag{6.4}$$

Substituting (6.4) into (1.1) and (6.3), we obtain an overdetermined system for the invariant functions $v(u)$ and $w(u)$ and the "superfluous" function $u(x, y, z)$. This system is compatible. Its general solution is specified by the following set of finite relations:

$$\begin{gathered} -u + v'^2(u) + w'^2(u) = 0, \\ -x + yv'(u) + zw'(u) + f'(u) = 0, \\ \varphi = -xu + yv(u) + zw(u) - f(u). \end{gathered} \tag{6.5}$$

The solution is determined with arbitrariness in two functions of one argument. For instance, we can arbitrarily define the functions $v(u)$ and $f(u)$. Then, the function $w(u)$ is found from the first equation of (6.5). The dependence $u(x, y, z)$ is implicitly determined by the second equation of (6.5). Substituting the functions obtained into the last relation of (6.5), we obtain the expression for the function $\varphi(x, y, z)$. The properties of the simple wave are described in [6] (the level surfaces of the simple wave are the characteristic planes). It should be noted that the simple wave (6.5) exists only in the domain of hyperbolicity of Eq. (1.1): $\varphi_x > 0$.

The simple wave for the resolving equation (2.5) is reduced to the invariant solution given by submodel 2.18.

6.2. *Double Wave.* In analyzing double waves, the theorem of reduction of double waves proved in [6] plays an important role. To formulate it, we introduce the notation $\lambda(\boldsymbol{x})$ and $\mu(\boldsymbol{x})$ for the double-wave parameters. We assume that the analysis of the double-wave equations yields a first-order subsystem, which is linear and uniform in terms of derivatives with respect to λ and μ:

$$A^i_\nu(\lambda, \mu)\lambda_i + B^i_\nu(\lambda, \mu)\mu_i = 0 \qquad (\nu = 1, \ldots, N). \tag{6.6}$$

Theorem 1. *If the number of independent equations in system* (6.6) *is* $N = 2n - 1$ (*n is the number of independent variables that are components of the vector* $\boldsymbol{x}$), *then the double wave is an invariant solution with respect to a certain subgroup of the group of translations of the space* $\mathbb{R}^n$ *prolonged by homothety.*

In the case considered, $n = 3$; therefore, by virtue of the theorem, the "prohibition of the fifth equation" is valid: if the analysis of the overdetermined system yields five equations of the form (6.6) for the "superfluous" functions, the corresponding solution is reduced to the invariant one.

With allowance for notation (6.1), we choose the solution representation in the form

$$u = u(v, w), \qquad v = v(x, y, z), \qquad w = w(x, y, z). \tag{6.7}$$

The function u is invariant, and the functions v and w are "superfluous." Substituting representation (6.7) into (1.1) and (6.3), we obtain four equations, which are written in the following form after certain transformations:

$$\begin{aligned} v_x &= \frac{v_y(1 + uu_v^2) + w_z(1 - uu_w^2)}{2uu_v}, \qquad & w_x &= \frac{v_y(1 - uu_v^2) + w_z(1 + uu_w^2)}{2uu_w}, \\ v_z &= \frac{v_y(1 - uu_v^2) + w_z(1 - uu_w^2)}{2uu_v u_w}, \qquad & w_y &= \frac{v_y(1 - uu_v^2) + w_z(1 - uu_w^2)}{2uu_v u_w}. \end{aligned} \tag{6.8}$$

For the double wave to be irreducible, the compatibility conditions (6.8) should be satisfied identically. Differentiating Eqs. (6.8) with respect to the variables x, y, and z, we obtain a system of 12 equations containing 12 derivatives of the functions v and w. The matrix of coefficients at the second derivatives has a rank equal to 10. Thus, there exist two linear combinations at which the second derivatives are eliminated from the system. One combination is an identity, i.e., it does not yield new equations for the first derivatives of the functions v and w. Eliminating the second derivatives with the help of the other linear combination, we obtain the equation satisfied in either of the two cases:

$$\text{(a)} \quad ((1 - uu_v^2)v_y + (1 - uu_w^2)w_z)^2 = 4u^2u_v^2u_w^2 v_y w_z; \tag{6.9}$$

$$\text{(b)} \quad (1 - uu_w^2)u_{vv} + (1 - uu_v^2)u_{ww} + 2uu_v u_w u_{vw} = 0. \tag{6.10}$$

In case (a), we have the fifth equation linearly relating the derivatives of the functions v and w. By virtue of the theorem, this means reduction of the solution to the invariant one. Equation (6.10) [case (b)] yields the condition on the invariant function $u(v,w)$ only. The fifth equation for the "superfluous" functions does not emerge; therefore, the solution is not reduced. We check the involution of system (6.8), (6.10). The Kartan characters [8] for this system are $\sigma_1 = 2$, $\sigma_2 = 0$, and $\sigma_3 = 0$. The Kartan number is $Q = 2$. Among all second derivatives of the "superfluous" functions u and v, only two are free by virtue of the prolonged system (6.8). If Eq. (6.10) is valid, the prolonged system does not impose additional restrictions on the first derivatives. Thus, the Kartan criterion is satisfied, and system (6.8) is in involution. Its solution is determined with arbitrariness in two functions of one argument.

We integrate the resultant system. Note, Eqs. (6.8) are equivalent to the following equations:

$$\varphi_x = u(\varphi_y, \varphi_z), \qquad (1 - uu_v^2)\varphi_{yy} + (1 - uu_w^2)\varphi_{zz} = 2uu_v u_w \varphi_{yz}. \tag{6.11}$$

The second equation of (6.11) is obtained by eliminating the derivatives φ_x and φ_{xx} from (1.1) by using the first equation of (6.11). The first equation of (6.11) is integrated by the Cauchy method [9]. In the parametric form, its solution is determined by the following formulas:

$$\begin{gathered} \varphi = u(p_1, p_2)x + p_1 y + p_2 z + \psi^0(p_1, p_2), \\ \frac{\partial u}{\partial p_1}x + y + \frac{\partial \psi^0}{\partial p_1} = 0, \qquad \frac{\partial u}{\partial p_2}x + z + \frac{\partial \psi^0}{\partial p_2} = 0. \end{gathered} \tag{6.12}$$

Here p_1 and p_2 are parameters and $\psi^0(p_1, p_2)$ is an arbitrary function. Writing the second equation of (6.11) with allowance for (6.12), we obtain

$$\Big(1 - u\,\frac{\partial u}{\partial p_1}\Big)\frac{\partial p_1}{\partial y} + \Big(1 - u\,\frac{\partial u}{\partial p_2}\Big)\frac{\partial p_2}{\partial z} = u\,\frac{\partial u}{\partial p_1}\,\frac{\partial u}{\partial p_2}\Big(\frac{\partial p_1}{\partial z} + \frac{\partial p_2}{\partial y}\Big). \tag{6.13}$$

Calculating the derivatives $(p_i)_y$ and $(p_i)_z$, substituting them into Eq. (6.13), and splitting the resultant expression with respect to the variable x, we obtain two equations. One of them determines the function u and coincides with Eq. (6.10) obtained in analyzing the compatibility conditions. The second equation is a restriction for the function $\psi^0(p_1, p_2)$:

$$\Big(1 - u\,\frac{\partial u}{\partial p_2}\Big)\frac{\partial \psi^0}{\partial p_1^2} + \Big(1 - u\,\frac{\partial u}{\partial p_1}\Big)\frac{\partial \psi^0}{\partial p_2^2} + 2u\,\frac{\partial u}{\partial p_1}\,\frac{\partial u}{\partial p_2}\,\frac{\partial \psi^0}{\partial p_1\,\partial p_2} = 0. \tag{6.14}$$

Thus, to construct the solution of the double-wave type, one has to determine the function u by solving Eq. (6.10) and then find the function ψ^0 as a solution of the linear equation (6.14). The sought potential $\varphi(x, y, z)$ is determined parametrically by formulas (6.12). The arbitrariness in the solution is four functions of one argument.

In contrast to the constructed solution, the double wave for the resolving equation (2.5) is reducible. Indeed, differentiation of the equations for the functions u, v, and w and elimination of the second derivatives in this case always yield the fifth equation linearly relating the first derivatives of v and w. This means reduction to the invariant solution by virtue of the theorem given above.

7. Result of Constructing the Optimal System ΘL_6. The normalized optimal system of subalgebras ΘL_6 is described in Table 5. The representatives of the optimal system are enumerated by a pair of numbers, the first one being the dimension r and the second one being the number N. The first column of Table 5 shows the number of the representative in a given dimension, and the second column contains the basis operators of the subalgebra (instead of the basis operators X_i, only their numbers i are given). The third column gives a reference (in the format "dimension.number") to the subalgebra that is a normalizer in L_6. If the representative of the optimal system indicated as a normalizer contains arbitrary parameters, their values are listed in the superscript in the alphabetical order (α, β). Thus, for the subalgebra $L_{1.1}$, we have the number $3.3^{0,0}$ in the third column. This means that its normalizer is the subalgebra $\{X_1, X_4, X_5\}$ (representative $L_{3.3}$, in which one should choose $\alpha = \beta = 0$). The sign "=" indicates a self-normalized subalgebra. The fourth column gives the list of finite invariants for each subalgebra. Since algebras (2.1) and (2.6) are isomorphic, the invariants are given in terms of the functions a and φ simultaneously. In some subalgebras, the invariants are written in the polar coordinates in the plane (y, z): $y = r\cos\theta$, $z = r\sin\theta$. Subalgebras of dimensions 4, 5, and 6 have no finite invariants.

TABLE 5

Representative number	Basis	Nor_{L_6}	Invariants
	$r = 6$		
1	$1, 2, 3, 4, 5, 6$	$= 6.1$	
	$r = 5$		
1	$1, 2, 4, 5, 6$	$= 5.1$	
2	$1, 2, 3, 4 + \alpha 6, 5 + \beta 6$	6.1	
3	$1, 2, 3, 4 + \alpha 5, 6$	6.1	
4	$1, 2, 3, 5, 6$	6.1	
	$r = 4$		
1	$1, 4, 5, 6$	$= 4.1$	
2	$2, 3, 4 + \alpha 6, 5 + \beta 6;\ \alpha^2 + \beta^2 \neq 0$	5.1	
3	$3, 4, 4, 5$	6.1	
4	$2, 3, \alpha 1 + 4, 1 + 5$	$5.2^{0,0}$	
5	$2, 3, 1 + 4, 5$	$5.2^{0,0}$	
6	$2, 3, 4 + \alpha 5, 6$	5.1	
7	$2, 3, 5, 6$	$= 4.7$	
8	$2, 3, 5, 6$	5.1	
9	$1, 2, 3, 4 + \alpha 5 + \beta 6$	6.1	
10	$1, 2, 3, 5 + \alpha 6$	6.1	
11	$1, 2, 3, 6$	6.1	
	$r = 3$		
1	$4, 5, 6$	$= 3.1$	$r^2 x^{-2} a,\ r^2 x^{-3} \varphi$
2	$3, 5, 6$	$= 3.2$	$y^2 x^{-2} a,\ y^2 x^{-3} \varphi$
3	$1, 4 + \alpha 6, 5 + \beta 6$	4.1	$r^{2(1-\beta)}\, \mathrm{e}^{-2\alpha\theta}\, a,\ r^{2-3\beta}\, \mathrm{e}^{-3\alpha\theta}\, \varphi$
4	$1, 4 + \alpha 5, 6$	4.1	$r\, \mathrm{e}^{-\alpha\theta}$
5	$1, 5, 6$	4.1	θ
6	$2, 3, 4 + \alpha 5 + \beta 6;\ \beta \neq 0$	5.1	$x^{2(\alpha-\beta)/\beta} a,\ x^{(2\alpha-3\beta)/\beta} \varphi$
7	$2, 3, 4 + \alpha 5$	6.1	x
8	$2, 3, 1 + 4 + \alpha 5$	$5.2^{0,0}$	$\mathrm{e}^{2\alpha x}\, a,\ \mathrm{e}^{2\alpha x}\, \varphi$
9	$1, 3, 5 + \alpha 6$	4.7	$y^{2(1-\alpha)} a,\ y^{2-3\alpha} \varphi$
10	$1 + 2, 3, 5 + 6$	$= 3.10$	$a,\ (x-y)^{-1} \varphi$
11	$2, 3, 5 + \alpha 6;\ \alpha \neq 0$	5.1	$x^{2(1-\alpha)/\alpha} a,\ x^{(2-3\alpha)/\alpha} \varphi$
12	$2, 3, 5$	6.1	x
13	$2, 3, 1 + 5$	$5.2^{0,0}$	$\mathrm{e}^{2x}\, a,\ \mathrm{e}^{2x}\, \varphi$
14	$1, 3, 6$	5.4	y
15	$1, 3, 2 + 6$	4.11	$\mathrm{e}^{-2y}\, a,\ \mathrm{e}^{-3y}\, \varphi$
16	$2, 3, 6$	5.1	$x^{-2} a,\ x^{-3} \varphi$
17	$1, 2, 3$	6.1	$a,\ \varphi$
	$r = 2$		
1	$4 + \alpha 6, 5 + \beta 6;\ \alpha^2 + \beta^2 \neq 0$	3.1	$x r^{-\beta}\, \mathrm{e}^{-\alpha\theta},\ r^2 x^{-2} a,\ r^2 x^{-3} \varphi$
2	$4, 5$	4.1	$x,\ r^2 a,\ r^2 \varphi$
3	$\alpha 1 + 4, 1 + 5$	$3.3^{0,0}$	$r\, \mathrm{e}^{\alpha\theta - x},\ r^2 a,\ r^2 \varphi$
4	$1 + 4, 5$	$3.3^{0,0}$	$x - \theta,\ r^2 a,\ r^2 \varphi$
5	$4 + \alpha 5, 6$	3.1	$r\, \mathrm{e}^{-\alpha\theta},\ r^2 x^{-2} a,\ r^2 x^{-3} \varphi$
6	$5, 6$	3.1	$\theta,\ r^2 x^{-2} a,\ r^2 x^{-3} \varphi$
7	$1, 4 + \alpha 5 + \beta 6$	4.1	$r\, \mathrm{e}^{-\alpha\theta},\ \mathrm{e}^{2(\alpha-\beta)\theta}\, a,\ \mathrm{e}^{(2\alpha-3\beta)\theta}\, \varphi$
8	$1 + 2, 5 + 6$	$= 2.8$	$(x-y)/z,\ a,\ z^{-1} \varphi$
9	$3, 5 + \alpha 6;\ \alpha \neq 0$	3.2	$x y^{-\alpha},\ y^2 x^{-2} a,\ y^2 x^{-3} \varphi$

TABLE 5 (Final)

Representative number	Basis	Nor_{L_6}	Invariants
10	$3, 5$	4.7	$x,\ y^2a,\ y^2\varphi$
11	$3, 1+5$	3.9^0	$x-\ln\lvert y\rvert,\ y^2a,\ y^2\varphi$
12	$1, 5+\alpha 6$	4.1	$\theta,\ r^{2(1-\alpha)}a,\ r^{2-3\alpha}\varphi$
13	$1, 3$	5.4	$y,\ a,\ \varphi$
14	$3, 6$	4.8	$y,\ x^{-2}a,\ x^{-3}\varphi$
15	$3, 2+6$	3.16	$x\,\mathrm{e}^{-y},\ x^{-2}a,\ x^{-3}\varphi$
16	$1, 6$	6.1	$y,\ z$
17	$1, 2+6$	4.11	$z,\ \mathrm{e}^{-2y}\,a,\ \mathrm{e}^{-3y}\,\varphi$
18	$1+2, 3$	4.10^1	$x-y,\ a,\ \varphi$
19	$2, 3$	6.1	$x,\ a,\ \varphi$
		$r = 1$	
1	$1+4+\alpha 5$	$3.3^{0,0}$	$x-\theta,\ r\,\mathrm{e}^{\alpha\theta},\ r^2a,\ r^2\varphi$
2	$4+\alpha 5+\beta 6;\ \beta\neq 0$	3.1	$rx^{-\alpha/\beta},\ r\,\mathrm{e}^{-\alpha\theta},\ r^2x^{-2}a,\ r^2x^{-3}\varphi$
3	$4+\alpha 5$	4.1	$x,\ r\,\mathrm{e}^{-\alpha\theta},\ r^2a,\ r^2\varphi$
4	$1+5$	$3.3^{0,0}$	$x-\ln\lvert r\rvert,\ \theta,\ r^2a,\ r^2\varphi$
5	$5+\alpha 6;\ \alpha\neq 0$	3.1	$\theta,\ xr^{-\alpha},\ r^{2(1-\alpha)}a,\ r^{2-3\alpha}\varphi$
6	5	4.1	$x,\ \theta,\ r^2a,\ r^2\varphi$
7	$2+6$	3.16	$z,\ x\,\mathrm{e}^{-y},\ \mathrm{e}^{-2y}\,a,\ \mathrm{e}^{-3y}\,\varphi$
8	6	5.1	$y,\ z,\ x^{-2}a,\ x^{-3}\varphi$
9	$1+2$	4.10^1	$x-y,\ z,\ a,\ \varphi$
10	3	5.4	$x,\ y,\ a,\ \varphi$
11	1	6.1	$y,\ z,\ a,\ \varphi$

Conclusions. The use of the group stratification for analyzing invariant solutions of the Kármán–Guderley equation allowed us to avoid classification of subalgebras for the infinite-dimensional admissible Lie algebra. The class of invariant solutions became wider than the one that could be obtained directly from Eq. (1.1) without integration of the automorphic system. The possibility of existence of the solution continuous in the entire space in submodels with a linear dependence of the independent variable on the polar angle is investigated. A partially invariant solution of the double-wave type is constructed. This solution is represented in a parametric form and is determined with arbitrariness in four functions of one argument.

Further construction of exact solutions of the Kármán–Guderley equation is possible in the case of a systematic study of partially invariant solutions generated by subalgebras $M \in \Theta L_6$, $\dim M \geqslant 3$. In addition, physical interpretation of the solutions obtained in the present work is needed.

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